Impulse and theorem of momentum 冲量和动量定理:

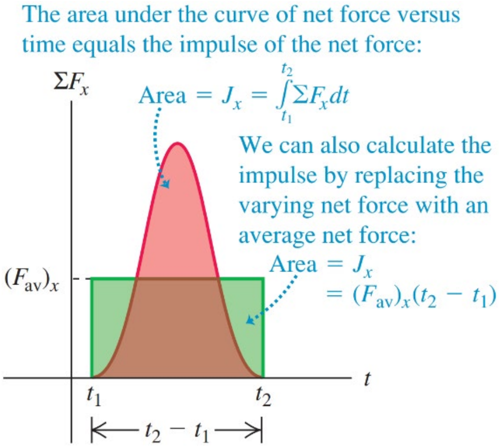
Momentum 动量:

$$\vec{P} = m\vec{v}$$

Impulse 冲量:

$$\vec{J} = \sum \vec{F}\,\Delta t = \sum \vec{F}\,(t_2 - t_1)$$

$$\vec{J} = \int \sum \vec{F}\,\mathrm{d}t$$



Impulse-Momentum Theorem

$$\vec{J} = \vec{P}_2 - \vec{P}_1 \quad \text{or}$$

$$\int_{t1}^{t2} \sum \vec{F}\,dt = \int_{\vec{P}_1}^{\vec{P}_2} d\vec{P} = \vec{P}_2 - \vec{P}_1$$

Average force $\vec{F}_{av}$: $\quad \vec{J} = \vec{F}_{av}(t_2 - t_1)$

Example: A *2kg* object is throwing toward a wall. During the collision with the wall lasting from $t = 0$ to $t = 0.2s$, the force acting on the object is given by the equation $\vec{F} = 300\mathrm{t}(0.2 - \mathrm{t})\vec{\imath}$ (N)

(a) The impulse that the force acts on the object during the collision is:

$$\vec{J} = \int \sum \vec{F}\,\mathrm{d}t = \int_0^{0.2} 300\mathrm{t}(0.2 - \mathrm{t})\vec{\imath}\,dt = (30t^2 - 100t^3)_0^{0.2} = 0.4\ \ (kgm/s)$$

(b) The average force on the object is:

$$\vec{F}_{av} = \frac{\vec{J}}{t_2 - t_1} = \frac{0.4}{0.2 - 0} = 2\ \ (N)$$

Exercise 11: A bullet shoot out of a gun at $v_0\ (m/s)$. While the bullet accelerates in the barrel of the gun, the total force that is applied on it is

$$F = a - bt \qquad (a, b \text{ are constant and } t \text{ is in second})$$

(a) Suppose the total force that the bullet is subjected to is zero upon the exit of the gun, calculate the total time that the bullet takes to run through the barrel.

$F = a - bt = 0\ \Rightarrow t = a/b$

(b) Calculate the momentum of the bullet.

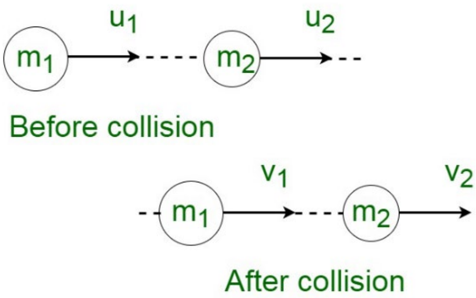
$$\int_0^P dP = \int_0^t Fdt = \int_0^{a/b} (a - bt)dt = \left(at - b\frac{t^2}{2}\right)_0^{a/b} = \frac{a^2}{2b}$$

(c) Calculate the mass of the bullet.

$$P = mv_0 = \frac{a^2}{2b} \Rightarrow m = \frac{a^2}{2bv_0}$$

Collision in one dimension(Central impact) 一维碰撞(对心碰撞)

General Case:



$$m_1u_1 + m_2u_2 = m_1v_1 + m_1v_2$$

Coefficient of restitution e: a measure of the elasticity of a collision. It is defined as the ratio of the speed of recession to the speed of approach.

$$\boldsymbol{e} = \frac{v_2 - v_1}{u_1 - u_2}$$

$$\Rightarrow \begin{cases} v_1 = \dfrac{m_1 - em_2}{m_1 + m_2}u_1 + \dfrac{(1+e)m_2}{m_1 + m_2}u_2 \\ v_2 = \dfrac{(1+e)m_1}{m_1 + m_2}u_1 - \dfrac{em_1 - m_2}{m_1 + m_2}u_2 \end{cases}$$

此公式不用背

Kinetic energy lost during the collision:

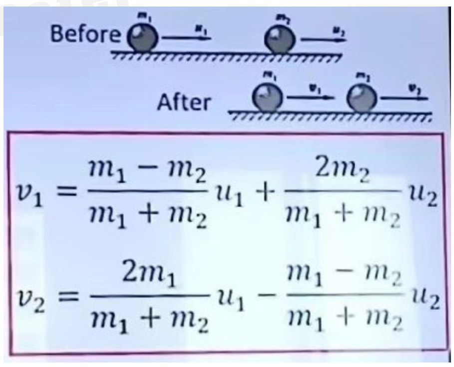
$$\Delta E_{k-lost} = E_{ki} - E_{kf} = \frac{1}{2}(1 - e^2)\frac{m_1m_2}{m_1 + m_2}(u_1 - u_1)^2$$

$e = 0$: Perfectly inelastic collision. ΔE_{k-lost} get the maximum value.

$e = \dfrac{v_2 - v_1}{u_1 - u_2} = 1$: **Elastic collision**. $\Delta E_{k-lost} = 0$

- In elastic collisions, the speed of separation equals the speed of approach.
- The Kinetic energy of the system is the same before and after the collision.

Some special situation in elastic collision



- $m_1 = m_2 \ \Rightarrow \ v_1 = u_2, v_2 = u_1$

  The velocities are swapped

- $m_1 \ll m_2$ and $u_2 = 0 \ \Rightarrow \ v_1 \approx -u_1, v_2 \approx 0$

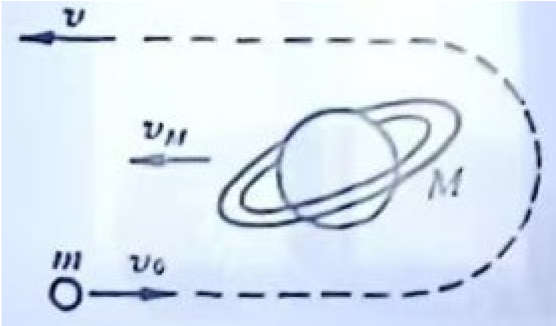
  The first object rebounds almost at the same speed

- $m_1 \gg m_2$ and $u_2 = 0 \ \Rightarrow \ v_1 \approx u_1, v_2 \approx 2u_1$

  The first object almost keeps the same velocity and the velocity of the second object is twice of the first one.

$0 < e < 1$: Inelastic collision.

Example: The sling shot effect is used to accelerate a spacecraft in a planetary flyby. The speed of Saturn relative to the Sun is $u_M = 9.6km/s$. A spacecraft with speed $u_0 = 10.4km/s$ relative to the Sun moves towards the Saturn. The spacecraft moves around and then leaves the Saturn. Calculate the speed of the spacecraft relative to the Sun when it leaves the Saturn.

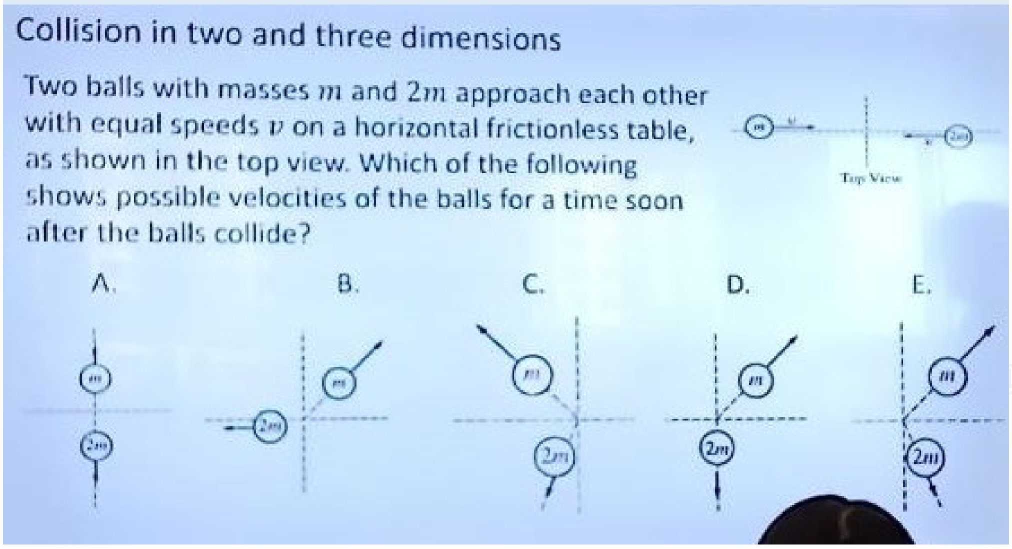


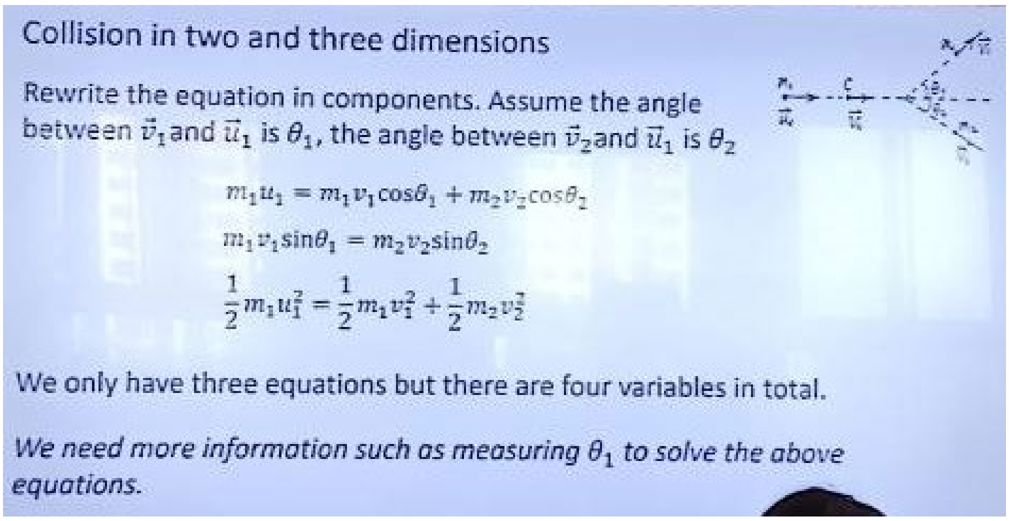
The entire procedure can be regards as elastic collision.

$$\left.\begin{aligned} Mu_M + mu_0 &= Mv_M + mv_f \\ \frac{1}{2}mu_0^2 + \frac{1}{2}mu_M^2 &= \frac{1}{2}mv_f^2 + \frac{1}{2}mv_M^2 \end{aligned}\right\} \Rightarrow v_f = \frac{2Mu_M - (M-m)u_0}{M+m}$$

Since $M \gg m \Rightarrow v_f = 2u_M - u_0 = 29.6\ (km/s)$

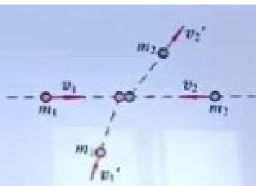
Collision in two and three dimensions 二三维碰撞





Collision in two and three dimensions

If the collision is not central impact, the situation will be complicated. It is a two dimensional even three dimensional problem. We just consider some simple cases here.



- ➢ Perfectly inelastic collision:

$$m_1\vec{u}_1 + m_2\vec{u}_2 = (m_1 + m_2)\vec{v}_f \Longrightarrow \vec{v}_f = \frac{m_1\vec{u}_1 + m_2\vec{u}_2}{m_1 + m_2}$$

- ➢ Elastic collision with $\vec{u}_2 = 0$: It is a two dimensional problem.

Conservation of linear momentum: $m_1\vec{u}_1 = m_1\vec{v}_1 + m_2\vec{v}_2$

Conservation of kinetic energy: $\frac{1}{2}m_1u_1^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$

Motion of object with varying mass

Discussion about the result: $v = v_r \ln\frac{m_0}{m_0 - m_f} - \frac{gm_f}{\alpha}$

- If we ignore all external forces:

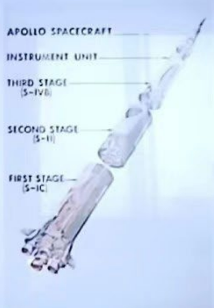
$$v = v_r \ln\frac{m_0}{m_0 - m_f} \Longrightarrow \text{Rocket equation}$$

- If we want to accelerate the rocket to a larger speed:

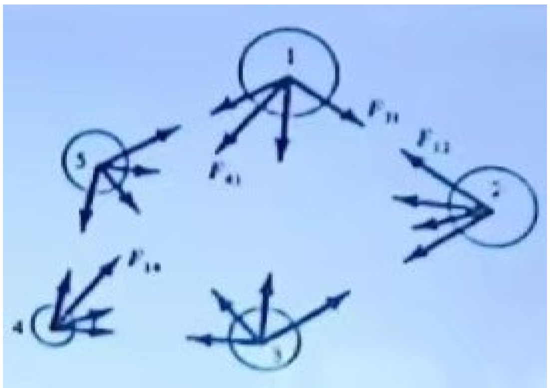
Multi-stage rocket: Using two or more rocket stages.

➢ For a two stage rocket:

$$v = v_r\left(\ln\frac{m_0}{m_0 - m_{f1}} + \ln\frac{m_1}{m_1 - m_{f2}}\right) = v_r \ln\left(\frac{m_0}{m_0 - m_{f1}} \cdot \frac{m_1}{m_1 - m_{f2}}\right)$$



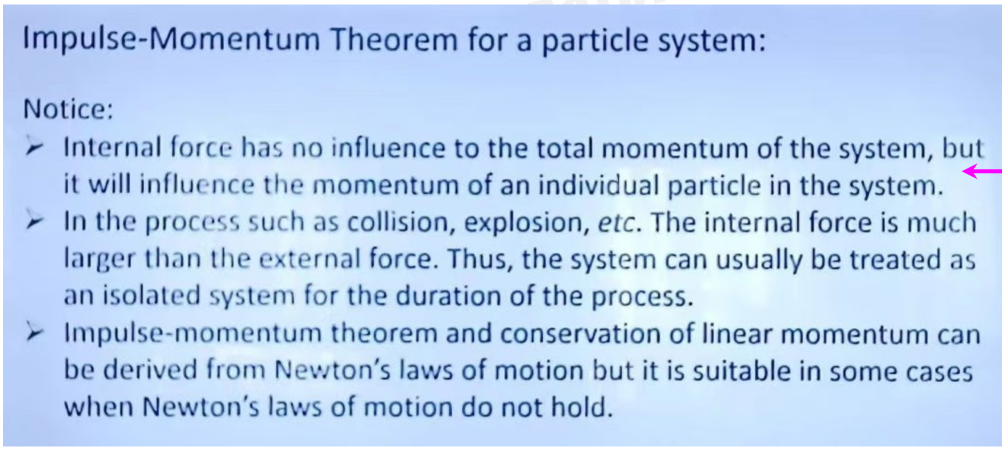
Particle System 质点系统



Considering a system consists of n particles, if the system is isolated: **The total momentum for an isolated system is conserved 孤立系统动量守恒.**

If the system is open and external force act on particles: **The change rage of total linear momentum of a system equals to the sum of all external force act on the system(内力不改变动量，只有外力改变动量).**

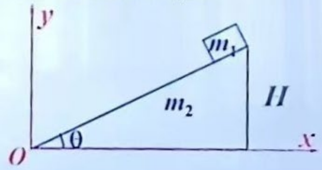
Summarize:
For an isolated system: $\frac{d\vec{p}_t}{dt} = 0 \Rightarrow \vec{p}_t = constant$ **Conservation of linear momentum 动量守恒定理**
For an open system: $\frac{d\vec{p}_t}{dt} = \vec{F}_{net-ext} \Rightarrow \Delta\vec{p}_t = \int_{t_1}^{t_2} \vec{F}_{net-ext} dt = \vec{I}_{net-ext}$
Notice: ● Impulse-momentum theorem is only suitable for **inertial reference frame.**   ● Linear momentum is determined by the state of the system.   ● Impulse is related to the process.   ● The equation is a vector equation, the net external force in any direction is zero, then the total linear momentum in that direction is conserved.



内力不改变系统动量

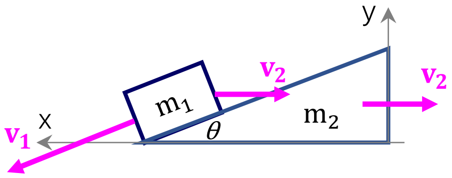
Impulse-Momentum Theorem for a particle system:

A small block of mass m_1 slides at rest down from the top of a slope inclined at an angle θ to the horizontal, the mass of slope is m_2 and the height of the slope is H as shown. All of the surfaces are smooth. Try to determine the displacement of the slope when block reaches the bottom of the slope.



Solution 1：动量守恒定律+机械能守恒定律

- 设m_2的速度为$\vec{v}_2$
- 设m_1相对于m_2的速度为$\vec{v}_1$
- m_1相对于地面的速度为$\vec{v}_1+\vec{v}_2$



1. 求m_1相对地面的速度

$$\begin{cases} x: & v_1\cos\theta - v_2 \\ y: & v_1\sin\theta \end{cases}$$

2. x 方向动量守恒

$$m_1(v_1\cos\theta - v_2) - m_2 v_2 = 0 \qquad (1)$$

3. 能量守恒

$$m_1 gH = \frac{1}{2}m_1[(v_1\cos\theta - v_2)^2 + (v_1\sin\theta)^2] + \frac{1}{2}m_2 v_2^2 \qquad (2)$$

4. 数学狂算

(1)和(2)联立，求出v_1^2
$v_1^2 = 2a_1(H/\sin\theta)$求出$a_1$
v_1和a_1求出 t
$m_1(a_1\cos\theta - a_2) = m_2 a_2$求出$a_2$
a_2和 t 求出x_2

$$(1) \Rightarrow v_2 = \frac{m_1\cos\theta}{m_1+m_2}v_1 \quad \text{plug into (2)}$$

$$m_1 gH = \frac{1}{2}m_1\left[\left(v_1\cos\theta - \frac{m_1}{m_1+m_2}v_1\cos\theta\right)^2 + (v_1\sin\theta)^2\right] + \frac{1}{2}m_2\left(\frac{m_1}{m_1+m_2}v_1\cos\theta\right)^2$$

$$m_1 gH = \frac{1}{2}m_1\left[\left(\frac{m_2}{m_1+m_2}\right)^2(v_1\cos\theta)^2 + (v_1\sin\theta)^2\right] + \frac{1}{2}m_2\left(\frac{m_1}{m_1+m_2}v_1\cos\theta\right)^2$$

$$2m_1 gH = m_1\left[\left(\frac{m_2}{m_1+m_2}\right)^2\cos^2\theta + \sin^2\theta\right]v_1^2 + m_2\left(\frac{m_1}{m_1+m_2}\cos\theta\right)^2 v_1^2$$

$$2m_1 gH = \left\{\left[m_1\left(\frac{m_2}{m_1+m_2}\right)^2 + m_2\left(\frac{m_1}{m_1+m_2}\right)^2\right]\cos^2\theta + m_1\sin^2\theta\right\}v_1^2$$

$$2m_1 gH = \{[m_1+m_2]\cos^2\theta + m_1\sin^2\theta\}v_1^2$$

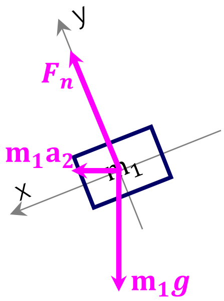
$$v_1^2 = \frac{2m_1 gH}{\{[m_1+m_2]\cos^2\theta + m_1\sin^2\theta\}} = 2a_1\left(\frac{H}{\sin\theta}\right)$$

$$a_1 = \frac{m_1 g\sin\theta}{(m_1+m_2)\cos^2\theta + m_1\sin^2\theta}$$

继续算下去太复杂了。

Solution 2: 非惯性系中的惯性力 Inertial forces in non-inertial frames

1. 对m_1放在非惯性系m_2中讨论(使用惯性力)



- Let the acceleration of m_2 be a_2 and ground is the reference frame. (inertial frame)
- Let a be the acceleration of m_1 with respect to m_2 (non-inertial frame)[1]

F_n: *Normal force*

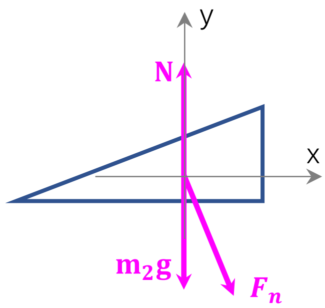
m_1g: *gravity*

m_1a_0: *pseudo* ← 非惯性系中的惯性力[1]

$$\begin{cases} m_1a_2\cos\theta + m_1g\sin\theta = m_1a \\ F_n + m_1a_2\sin\theta = m_1g\cos\theta \end{cases} \quad (1)$$

$$\Rightarrow a = a_2\cos\theta + g\sin\theta \quad (2)$$

2. 对m_2放在惯性系中讨论



- N: *Normal force*
- m_2g: *gravity*
- F_n: *Normal to the incline (by the block m_1)*

$$F_n\sin\theta = m_2a_2 \;\Rightarrow\; F_n = m_2a_2/\sin\theta \quad (3)$$

Plug (3) into (1)

$$\frac{m_2a_2}{\sin\theta} + m_1a_2\sin\theta = m_1g\cos\theta$$

$$\Rightarrow a_2 = \frac{m_1g\sin\theta\cos\theta}{m_2 + m_1\sin^2\theta} \quad (4)$$

3. 求出m_1相对于m_2的加速度 a

Plug (4) into (2)

$$a = \frac{m_1g\sin\theta\cos\theta}{m_2 + m_1\sin^2\theta}\cos\theta + g\sin\theta = \frac{(m_1+m_2)g\sin\theta}{m_2 + m_1\sin^2\theta}$$

4. m_1在 slop 上的滑行距离 s 是

$$s = \frac{H}{\sin\theta} = \frac{1}{2}at^2 \Rightarrow t^2 = \frac{2s}{a} = \frac{2H}{a\sin\theta}$$

5. Slop 本身滑行的距离x_2是

$$x_2 = \frac{1}{2}a_2t^2 = \frac{1}{2}\left(\frac{m_1\sin\theta\cos\theta}{m_2 + m_1\sin^2\theta}g\right)\frac{2H}{a\sin\theta}$$

$$= \frac{g}{2}\left(\frac{m_1\sin\theta\cos\theta}{m_2 + m_1\sin^2\theta}\right)\frac{2H}{\sin\theta}\frac{m_2 + m_1\sin^2\theta}{(m_1+m_2)g\sin\theta}$$

$$= \frac{m_1\cot\theta}{m_1 + m_2}H$$

Key Point:

1. 分清m_1是在非惯性系中讨论，需要使用惯性力。但是m_2是在惯性系中讨论，不需要使用惯性力
2. 惯性力的大小是物体质量m_1和非惯性系加速度a_2的乘积，方向和a_2相反

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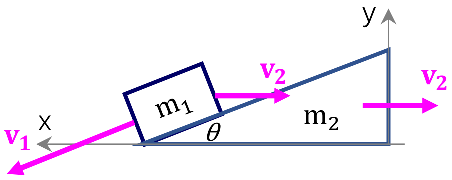
[1] https://en.wikipedia.org/wiki/Fictitious_force

Solution 3: 动量守恒定律直接应用于位移

- 设m_2的速度为$\vec{v}_2$
- 设m_1相对于m_2的速度为$\vec{v}_1$
- m_1相对于地面的速度为$\vec{v}_1+\vec{v}_2$



1. x 方向动量守恒

$$m_1(v_1\cos\theta - v_2) - m_2 v_2 = 0$$

$$\Rightarrow m_1\left(\int_0^t v_1\cos\theta\,dt - \int_0^t v_2 dt\right) - m_2\int_0^t v_2 dt = 0$$

$$\Rightarrow\ m_1(x_1\cos\theta - x_2) - m_2 x_2 = 0$$

其中x_1和x_2分别为m_1和m_2的位移

$$m_1(x_1\cos\theta - x_2) = m_2 x_2$$

$$m_1 x_1\cos\theta = m_1 x_2 + m_2 x_2 \qquad (1)$$

2. 分析相对运动

水平方向上m_1相对m_2运动的距离是$x_1\cos\theta$

$$H = \frac{x_1}{\sin\theta} \Rightarrow\ x_1 = \frac{H}{\sin\theta} \quad \text{plug into (1)}$$

$$m_1\frac{\mathrm{H}}{\sin\theta}\cos\theta = (m_1+m_2)x_2$$

$$x_2 = \frac{m_1}{m_1+m_2}\frac{\mathrm{H}\cos\theta}{\sin\theta}$$

Key Point:

1. 动量守恒定律不仅对于 v 是适用的，对于 x 也是适用的。
   x 方向受力为 0
   ⇒ x 方向动量守恒
   $\Rightarrow\ m_1u_1+m_2u_2=m_1v_1+m_2v_2$
   $\Rightarrow\ m_1x_{u1}+m_2x_{u2}=m_1x_{v1}+m_2x_{v1}$ (直接将速度替换为位移)
2. 注意相对运动的分析
   m_1相对m_2的速度是$\vec{v}_1\ \Rightarrow\ m_1$相对$m_2$的位移是$\vec{x}_1$
   m_1相对地面的速度是$\vec{v}_1+\vec{v}_2$

Center of mass 质心

Center of mass 质心:

For a system consists of n particles with mass m_i and position (x_i, y_i)

$$x_{cm} = \frac{m_1x_1 + m_2x_2 + \cdots}{m_1 + m_2 + \cdots} = \frac{\sum_i m_i x_i}{\sum_i m_i}$$

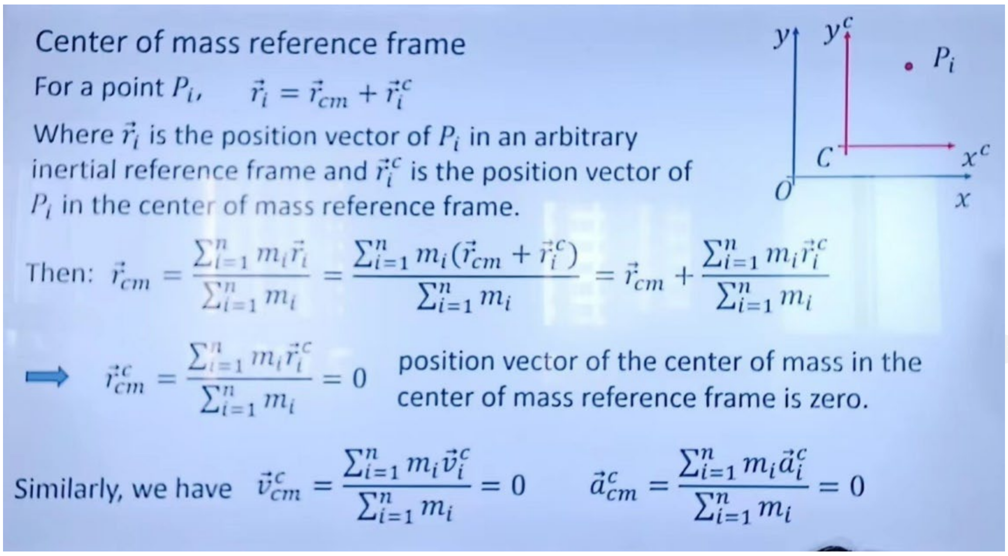
$$y_{cm} = \frac{m_1y_1 + m_2y_2 + \cdots}{m_1 + m_2 + \cdots} = \frac{\sum_i m_i y_i}{\sum_i m_i}$$

For a system with continuous mass distribution:

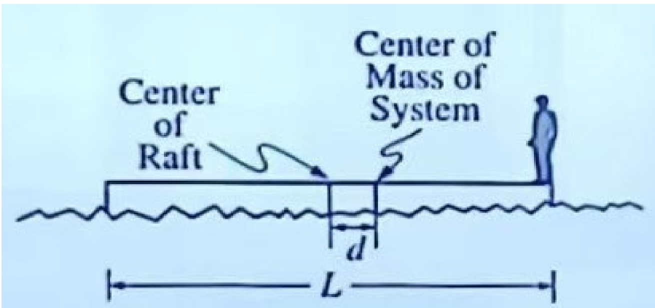
$$x_{cm} = \frac{\int x dm}{\int dm}$$

$$y_{cm} = \frac{\int y dm}{\int dm}$$

Center of mass reference frame 质心参考系:



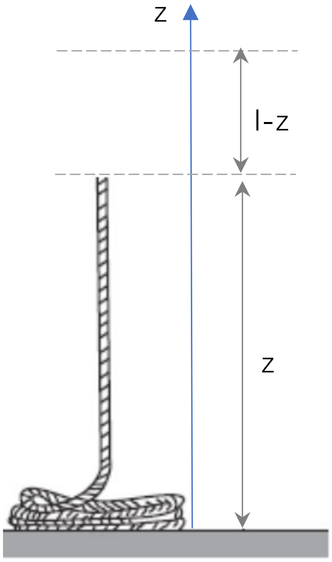
Example 1: A person is standing at one end of a uniform raft of length L that is floating motionless on water, as shown. The center of mass of the person-raft system is a distance d from the center of the raft. The person then walks to the other end of the raft. If friction between the raft and the water is negligible, how far does the draft move relative to the water?



$$x = 2d$$

此题和 p8 题目是否很类似？

Center of mass reference frame 质心参考系



A uniform soft rope with mass m and length l hangs vertically with its lower end just touching the ground. The rope is released from rest. Try to calculate the force of the ground acting on the rope when the remaining length in the air is z.

Momentum s 动量: $\vec{p}_{sys} = m\vec{v}_{cm}$

牛顿第二定律: $\sum \vec{F}_{ext} = m\vec{a}_{cm}$

$\lambda = \frac{m}{l}$

Center of mass location

$$z_{cm} = \frac{1}{m}\int_0^z \lambda z\,dz = \frac{1}{m}\int_0^z \frac{m}{l} z\,dz = \frac{z^2}{2l}$$

$$v_{cm} = \frac{dz_{cm}}{dt} = \frac{z}{l}\left(\frac{dz}{dt}\right)$$

$$a_{cm} = \frac{dv_{cm}}{dt} = \frac{1}{l}\left(\frac{dz}{dt}\right)^2 + \frac{z}{l}\left(\frac{d^2z}{dt^2}\right) = \frac{2g(l-z)}{l} - \frac{gz}{l} = 2g - 3g\frac{z}{l}$$

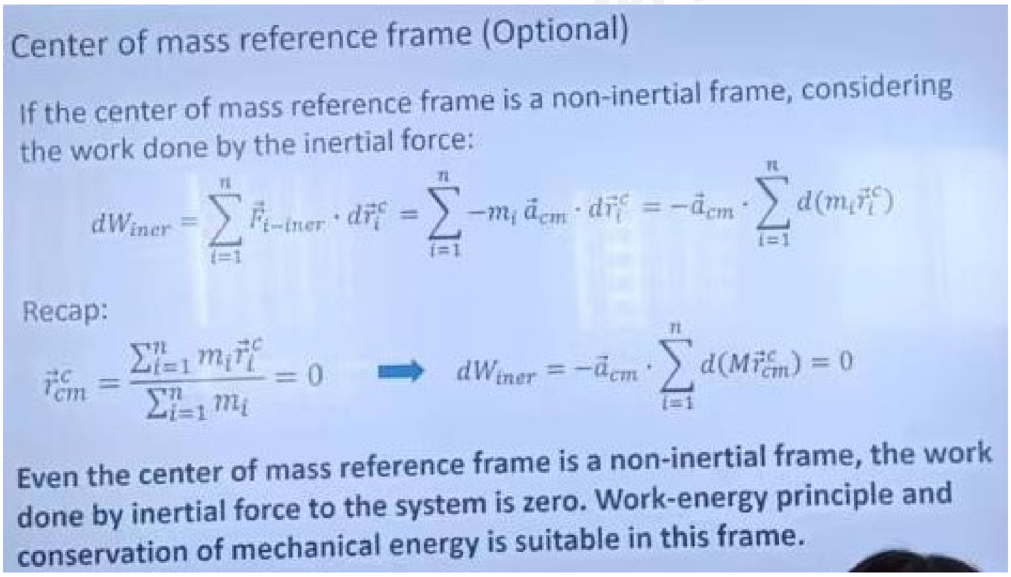
$$\frac{dz}{dt} = v_z = -\sqrt{2g(l-z)}$$

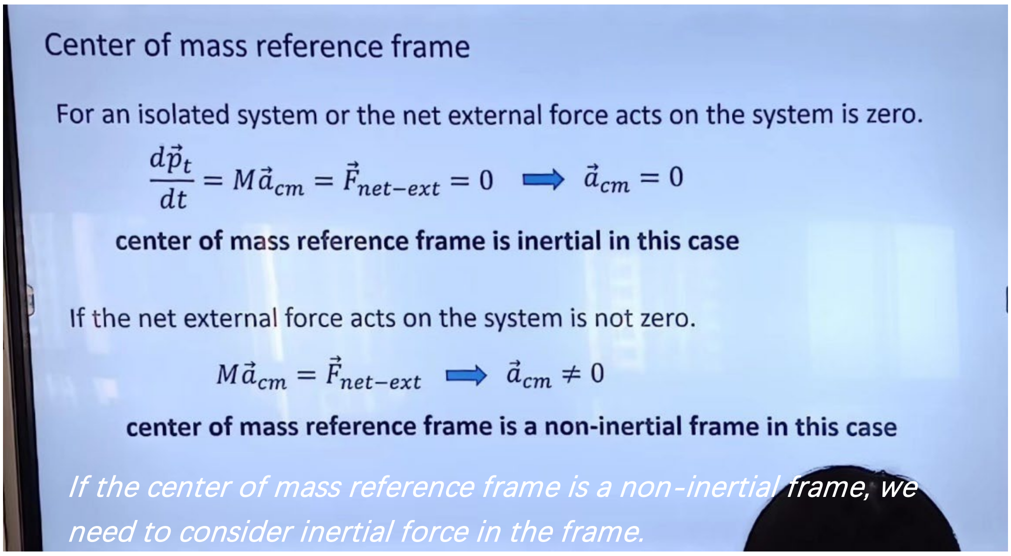
$$\frac{d^2z}{dt^2} = a_z = -g$$

此方法的关键点是写出v_z和a_z的表达式。代入上式
这种方法有点难。

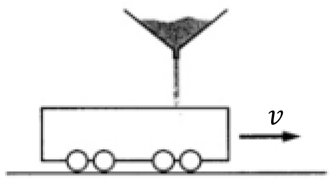
$$ma_{cm} = F - mg \Rightarrow F = ma_{cm} + mg = m\left(2g - 3g\frac{z}{l}\right) + mg = 3mg\frac{l-z}{l}$$

If the center of mass reference frame is a non-inertial frame(Optional)




Center of mass reference frame
For an isolated system or the net external force acts on the system is zero.
$\frac{d\vec{p}_t}{dt} = M\vec{a}_{cm} = \vec{F}_{net-ext} = 0 \Rightarrow \vec{a}_{cm} = 0$
center of mass reference frame is inertial in this case
If the net external force acts on the system is not zero.
$M\vec{a}_{cm} = \vec{F}_{net-ext} \Rightarrow \vec{a}_{cm} \neq 0$
center of mass reference frame is a non-inertial frame in this case
If the center of mass reference frame is a non-inertial frame, we need to consider inertial force in the frame.

Motion of Object with Varying Mass 变质量系统



Example: An open-top railroad car (initially empty and of mass m_0) rolls with negligible friction along a straight horizontal track and passes under the spout of a sand conveyor. When the car is under the conveyor, sand is dispensed from the conveyor in a narrow stream at a steady rate $dm/dt = C$ and falls vertically to the car. The car has initial speed v_0 and sand is filling it from time $t = 0$ to $t = T$.

(a) Calculate the mass m of the car plus the sand that it catches as a function of time t from 0<t<T.

Weight of sand: $dm = Cdt \Rightarrow \int_0^{m_s} dm = \int_0^T Cdt \Rightarrow m_s = Ct$

Total weight $m = m_s + m_0 = Ct + m_0$

(b) Determine the speed v of the car as a function of time t from 0<t<T.

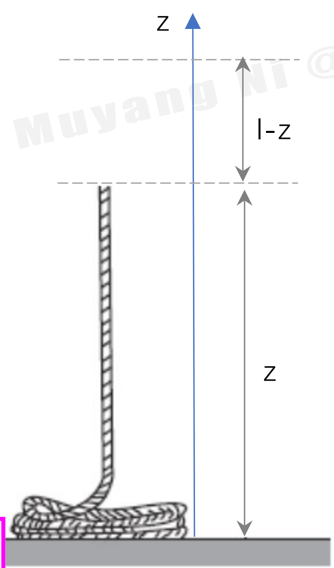
In the horizontal direction, as the friction is negligible, the Momentum of the car-sand system keeps constant:

$$P = m_0 v_0 = (Ct + m_0)v(t) \Rightarrow \mathrm{v(t)} = \frac{m_0 v_0}{m_0 + Ct}$$

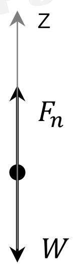
(c) If you want to keep the velocity of the car constant, what kind of force needs to act on the car.

To keep the sand-car system's velocity constant, an impulse must be applied to the sand:

$$\int_0^t Fdt = \int_0^m v_0 dm = \int_0^t v_0 Cdt \Rightarrow Ft = v_0 Ct \Rightarrow F = Cv_0$$



Example: A uniform soft rope with mass m and length l hangs vertically with its lower end just touching the ground. The rope is released from rest. Try to calculate the force of the ground acting on the rope when the remaining length in the air is z.



研究对象:已经落下的绳子+在时间 dt 内新落下的绳子

在时间 dt 内，新落下绳子的长度是 dz，其质量是 dm.

$$(F_n - W)dt = [0 - (-v)]dm$$

W 是落下绳子的质量

dm 可以通过时间 dt 求出

$$W = m\frac{l-z}{l}\mathrm{g}$$

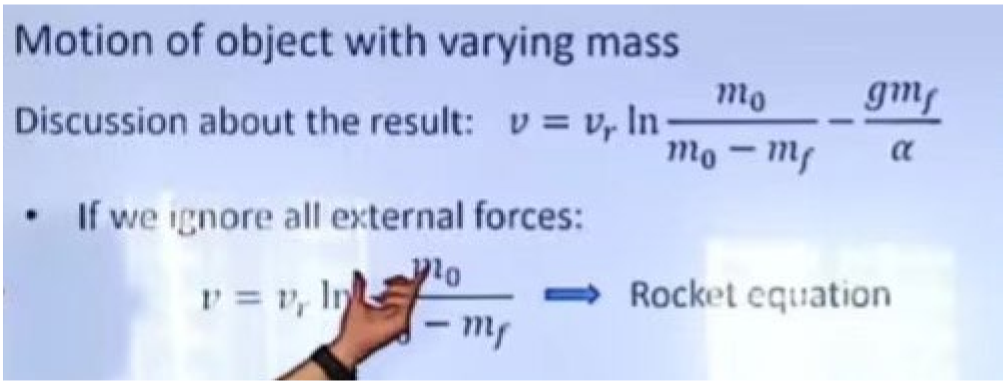
$$dm = \frac{m}{l}dz = \frac{m}{l}(vdt)$$

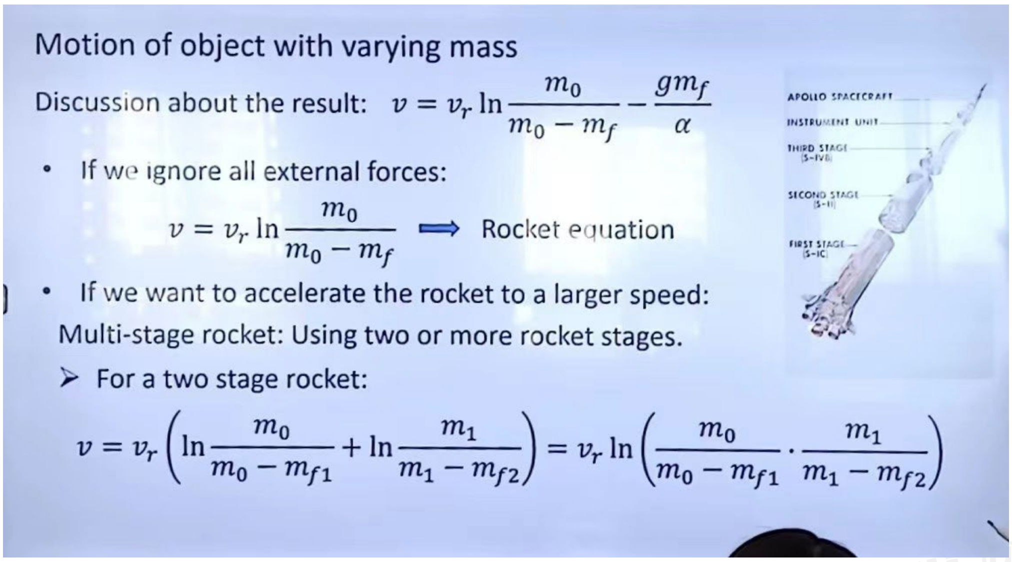
$$\left(F_n - m\frac{l-z}{l}\mathrm{g}\right)dt = vdm = v\frac{m}{l}(vdt) = v^2\frac{m}{l}dt \quad (1)$$

$v^2 = 2g(l-z)$, plug into (1) $\Rightarrow$ $F_n - m\frac{l-z}{l}\mathrm{g} = [2g(l-z)]\frac{m}{l} = 3mg\frac{l-z}{l}$

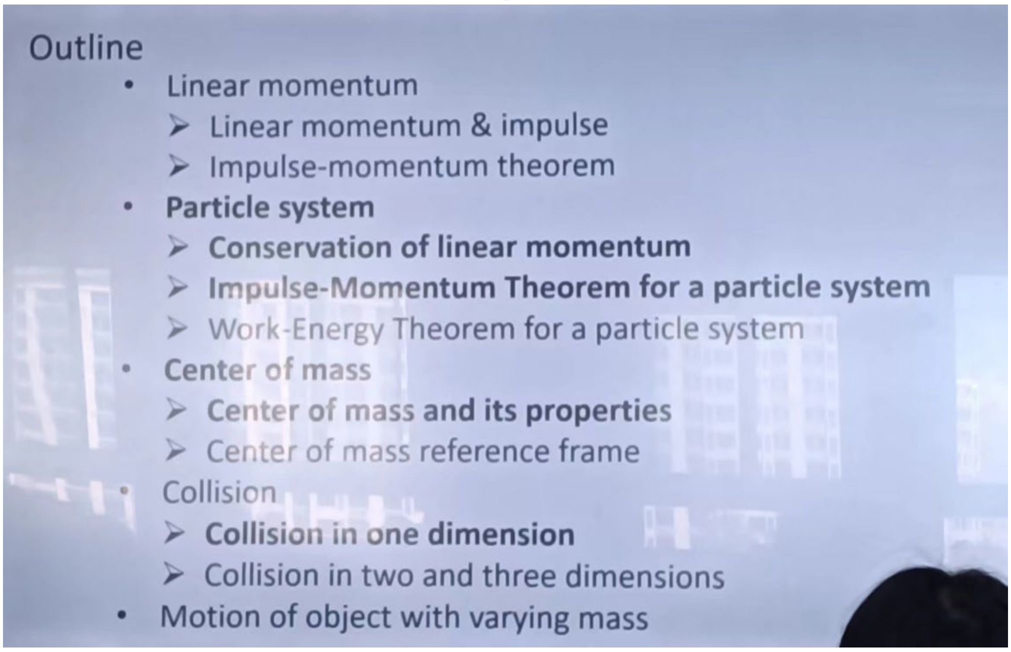
Key Point:

1. 选取的研究对象是已经落在秤上的绳子和时间 dt 内落下绳子的总和
2. $[0-(-v)]dm$ 此式中符号很容易出错。





Outline 总结



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